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Extended class of phenomenological universalities

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ABSTRACT

The phenomenological universalities (PU) are extended to include quantum oscillatory phenomena, coherence and supersymmetry. It will be proved that this approach generates minimum uncertainty coherent states of time-dependent oscillators, which in the dissociation (classical) limit reduce to the functions describing growth (regression) of the systems evolving over time. The PU formalism can be applied also to construct the coherent states of space-dependent oscillators, which in the dissociation limit produce cumulative distribution functions widely used in probability theory and statistics. A combination of the PU and supersymmetry provides a convenient tool for generating analytical solutions of the Fokker–Planck equation with the drift term related to the different forms of potential energy function. The results obtained reveal existence of a new class of macroscopic quantum (or quasi-quantum) phenomena, which may play a vital role in coherent formation of the specific growth patterns in complex systems.

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1. Introduction

The concept of PU introduced by Castorina, Delsanto, and Guiot (CDG) [1,2] concerns ontologically different systems, in which miscellaneous emerging patterns are described by the same mathematical formalism. Universality classes are useful for their applicative relevance and facilitate the cross fertilization among various fields of research, including physics, chemistry, biology, ecology, engineering, economics and social sciences [1–18]. This strategy is extremely important, especially for the export of ideas, models and methods developed in one discipline to another and vice versa. The PU approach is also a useful tool for investigation of the complex systems whose evolution is governed by nonlinear processes. Hence, this methodology can be employed [1] to obtain different functions of growth widely applied in actuarial mathematics, biology and medicine. In this work the research area is extended to include in the CDG scheme the quantum oscillatory phenomena, coherence and supersymmetry. In particular it will be proved that the CDG formalism is a hidden form of supersymmetry, which can be employed not only to produce macroscopic growth functions but also to construct quantum coherent states of the time- and space-dependent Morse [20] and Wei [21] oscillators. In the dissociation (classical) limit they reduce to the well-know Gompertz [22] and West–Brown–Enquist (WBE)-type [23] functions (e.g. logistic, exponential, Richards, von Bertalanffy) describing sigmoidal (S-shaped) temporal evolution or spatial distribution of subelements of complex systems. We shall also be concerned with a generalization of the CDG approach to include regression states which has not been consider in the original formulation of the CDG theory.

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2. Theory

According to the CDG theory, various degrees of nonlinearity appearing in the complex systems can be described and classified using the set of nonlinear equations [1]

$$\frac{d\psi(q)}{dq} - x(q)\psi(q) = 0, \qquad \frac{dx(q)}{dq} + \Phi(x) = 0. \tag{1}$$

Here, $q = u_t t$ denotes dimensionless temporal variable, u_t is a scaling factor, whereas $\Phi(x)$ stands for a generating function, which expanded into a series of x-variable (it slightly differs from the original CDG formulae) [1]

$$\Phi(x) = c_1(x + c_0/c_1) + c_2(x + c_0/c_1)^2 + \ldots \tag{2}$$

produces different functions of growth $\psi(q)$ for a variety of patterns emerging in the systems under consideration. To obtain their explicit forms a combination of Eqs. (1) is integrated generating the growth functions [1]

$$\psi(q) = \exp\left[-\int_x \frac{xdx}{\Phi(x)} + C\right] = \exp\left[\int_q x(q)dq + C\right] \tag{3}$$

for different powers $n = 1, 2, \ldots$ of the truncated series (2). The integration constant C can be calculated from a boundary condition $x(q = 0)$. For example for $x(0) = 1$, $c_0 = 0$, $c_1 = 1$, $\Phi(x) = x$ one gets the Gompertz function [22], whereas for $\Phi(x) = x + c_2 x_2$ the allometric WBE-type function [23] can be derived [1]

$$\psi(q)_G = \exp\left[(1 - \exp(-q)\right],$$
$$\psi(q)_W = \exp\left[1 + c_2 - c_2 \exp(-q)\right]^{1/c_2}. \tag{4}$$

Employing this approach the PU can be classified [1] as $U1$ ($n = 1$), $U2$ ($n = 2$) etc. with respect to the different levels of nonlinearity utilized by the complex systems during formation of the specific growth patterns. In subsequent works [9,10] the PU concept has been extended to include parameter c_1 and $c_0 = \Phi(x) = 0$. In the latter case, the solutions of Eqs. (1) and (3) take the form $x(q) = x(0)$, $\psi(q) = \exp[x(0)q]$, which for $q = u_t t$ represents $U0$ class of PU describing self-catalytic processes [10]. The solutions of Eqs. (1) and (3) obtained in the original CDG scheme take the form monotonic growth curves, with no allowance for oscillations, which are ubiquitous dynamic feature of the complex systems observed in nature. Oscillations are usually the results of the mutual interferences between a growing system and its surrounding, or between several competing processes appearing in the same system. To describe such phenomena Barberis et al. [11] employed a complex function, whose real and imaginary parts represent two phenotypic traits of the same organism. As the result a generalization of the Gompertz and WBE growth models has been obtained. The PU complex field formalism has been applied also by Delsanto et al. [9] in analyzing the evolution of the system depending on two variables driven by the set of coupled nonlinear differential equations (1). They reproduced main oscillatory features of the time-evolution curves belonging to the complex counterparts of $U1$ and $U2$ classes of PU. In another model proposed by Barberis and co-workers [12,13] interactive growth phenomena in biological and ecological systems are described using vector formulation of PU in the real space. In this way the joint growth of two or more interacting organisms as well as mutual influences that operate through environment modifications can be characterized without *ad hoc* formulated assumptions on the nature of the interactions. The vector universalities model has been also applied to describe the cancer growth viewed as the result of the competition between two or more cancer cell populations [14]. Recently, the classical oscillations have been also considered [17] in the CDG scheme generalized by Molski [19] employing only real variable. In this letter it will be proved that the PU classification scheme embraces not only classical but also quantum oscillatory phenomena, whose complete characteristics can be determined without the use of complex formalism. On the other hand, the proposed supersymmetric interpretation of the CDG theory is based on the two component (vector) functions build up with the growth and regression terms, hence we find here a some analogy to the Barberis and co-workers formalism [12–14]. However, it should be pointed out that the growth-regression states represent independent and uncorrelated in time processes, so the continuous transition of the growth phase to the regression state and *vice versa* is a genuine property of the systems under consideration.

3. Results

The CDG approach can be easily extended to include the space-dependent phenomena using spatial variable $q = u_r r$ in which u_r is a scaling factor. In this way one may generate in the CDG scheme the space-dependent sigmoidal Gompertz and WBE-type functions widely applied in a range of fields including e.g. probability theory and statistics where they are used to describe cumulative distribution of entities characterized by different spatial sizes [24]. In view of this the CDG formalism can describe not only temporal evolution of the complex systems represented by $\psi(t)$ but also the spatial distribution $\psi(r)$ of their subcomponents. In particular, the spatial version of the Gompertz function (4) has been found as a powerful descriptive tool for neuroscience where can be used as cumulative distribution curve correctly describing diameters of fibers in the olfactory nerves [24].

3.1. Regression

A detailed analysis of the CDG approach reveals that it does not take into account a very important phenomenon of regression (decay) appearing in biological, medical, demographic and economic systems. Such an effect appears, for example, under chemotherapeutic treatment of tumors subjected to cycle specific (or nonspecific) drugs causing regression of cancer whose growth (decay) is described by the Gompertz function [25]. Recent investigations in the field of neurology revealed also that the temporal Gompertz function of regression can be employed to describe time course of synaptic current or change the membrane conductance during voltage clamp of squid axon [24]. To include the regression phenomenon in the CDG scheme, Eqs. (1) and (3) should be modified to the form

$$\frac{d\psi(q)^\dagger}{dq} + x(q)\psi(q)^\dagger = 0, \tag{5}$$

$$\psi(q)^\dagger = \exp\left[\int_x \frac{xdx}{\Phi(x)} + C\right] = \exp\left[-\int_q x(q)dq + C\right], \tag{6}$$

which for $\Phi(x) = x$ and $\Phi(x) = x + c_2 x^2$ produce Gompertz and WBE-type functions of regression [24,25]

$$\psi(q)^\dagger_G = \exp\left[-(1 - \exp(-q)\right],$$
$$\psi(q)^\dagger_W = \exp\left[1 + c_2 - c_2 \exp(-q)\right]^{-1/c_2}. \tag{7}$$

It is easy to demonstrate that for $q \to \infty$ the functions $\psi(q)^\dagger_G \to \exp(-1)$, $\psi(q)^\dagger_W \to (1 + c_2)^{-1/c_2}$ diminish with q, hence they describe the regression states of the system under consideration. The regression states have been generated also in [18] by a proper choice of the parameters defining the growth functions or by applying the involuted Gompertz function derived by Molski [19]. In contrast to this approach Eq. (5) can be applied to derive regression states associated with all types of growth functions created in the CDG scheme independently of parameters defining them.

3.2. Supersymmetry

Analysis of Eqs. (1) and (5) reveals that they can be interpreted in the framework of time-dependent ($q = u_t t$) [26] or space-dependent ($q = u_r r$) [27] quantum supersymmetry (SUSYQM), used among others to construct coherent states of quantum oscillators and to obtain exact solutions of the Schrödinger equation for vibrating harmonic and anharmonic systems. In view of this, it is tempting to apply the CDG methodology to generate the coherent states of time- and space-dependent oscillators and compare them with those obtained using algebraic procedure [28,29]. To prove that mathematical formalism of PU is a hidden form of supersymmetry, lets differentiate growth equation (1) once with respect to q-coordinate and then rearrange the derived formulae to obtain the second order differential equation in a standard eigenvalue form

$$\frac{d^2\psi(q)}{dq^2}-\psi(q)\frac{dx(q)}{dq}-x(q)\frac{d\psi(q)}{dq}=$$
$$\left[-\frac{1}{2}\frac{d^2}{dq^2}+V(q)-\epsilon\right]\psi(q)=\left(\hat{H}-\epsilon\right)\psi(q)=0,$$
$$V(q)-\epsilon=\frac{1}{2}\left[x(q)^2+\frac{dx(q)}{dq}\right], \tag{8}$$

represents (with accuracy to multiplicative constant) the Riccati equation [26,27]. In similar manner one may derive

$$\frac{d^2\psi(q)^\dagger}{dq^2}+\psi(q)^\dagger\frac{dx(q)}{dq}+x(q)\frac{d\psi(q)^\dagger}{dq}=$$
$$\left[-\frac{1}{2}\frac{d^2}{dq^2}+V(q)^\dagger-\epsilon\right]\psi(q)^\dagger=\left(\hat{H}^\dagger-\epsilon\right)\psi(q)^\dagger=0,$$
$$V(q)^\dagger-\epsilon=\frac{1}{2}\left[x(q)^2-\frac{dx(q)}{dq}\right], \tag{9}$$

employing the regression equation (5). The quantity $x(q)$ appearing in Eqs. (1), (8) and (9) has a dual interpretation: in algebraic methods $+x(q)$ represents an anharmonic variable [29], whereas in SUSYQM, $-x(q)=W(q)$ stands for a superpotential [27], which permits rewriting the Hamilton operators appearing in Eqs. (8) and (9) in the terms of first-order annihilation and creation operators $\hat{A}$ and $\hat{A}^\dagger$

$$\hat{H}=\frac{1}{\sqrt{2}}\left[-\frac{d}{dq}+W(q)\right]\frac{1}{\sqrt{2}}\left[\frac{d}{dq}+W(q)\right]=\hat{A}^\dagger\hat{A},$$
$$\hat{H}^\dagger=\frac{1}{\sqrt{2}}\left[\frac{d}{dq}+W(q)\right]\frac{1}{\sqrt{2}}\left[-\frac{d}{dq}+W(q)\right]=\hat{A}\hat{A}^\dagger. \tag{10}$$

In SUSYQM operators $\hat{H}$ and $\hat{H}^\dagger$ form the two-component Hamiltonian [27]

$$\hat{H}_S=\begin{bmatrix}\hat{H} & 0\\ 0 & \hat{H}^\dagger\end{bmatrix} \tag{11}$$

including fermionic $\hat{H}$ and bosonic $\hat{H}^\dagger$ components, respectively. They are supersymmetric partners of each other and correspond to an isospectral pair of potentials $V(q)$ and $V(q)^\dagger$. In the CDG methodology fermionic and bosonic components can be associated with the growth $\psi(q)$ and regression $\psi(q)^\dagger$ functions, respectively. They are solutions of the supersymmetric isospectral equation

$$\hat{H}_S\Psi=\begin{bmatrix}\hat{H} & 0\\ 0 & \hat{H}^\dagger\end{bmatrix}\begin{bmatrix}\psi(q)\\ \psi(q)^\dagger\end{bmatrix}=\epsilon\begin{bmatrix}\psi(q)\\ \psi(q)^\dagger\end{bmatrix} \tag{12}$$

or can be calculated from Eq. (3) specified in the form well-known in SUSYQM [27]

$$\psi(q)=\exp\left[-\int_q W(q)dq+C\right],$$
$$\psi(q)^\dagger=\exp\left[+\int_q W(q)dq+C\right] \tag{13}$$

being supersymmetric versions of the CDG formulae (3) and (6). To demonstrate an apparent connection between CDG and SUSYQM formalisms, lets consider the Gompertzian system characterized by $x(0)=1$, $\Phi(x)=x$, $x(q)=\exp(-q)$ yielding Eqs. (8) and (9) in the explicit forms

$$\left\{-\frac{1}{2}\frac{d^2}{dq^2}+\frac{1}{8}\left[1-\exp[-(q-q_0)]\right]^2-\frac{1}{8}\right\}\psi(q)_G=0,$$
$$\left\{-\frac{1}{2}\frac{d^2}{dq^2}+\frac{1}{8}\left[1+\exp[-(q-q_0)]\right]^2-\frac{1}{8}\right\}\psi(q)_G^\dagger=0, \tag{14}$$

in which $q_0=\ln(2)$. One may also prove that Eqs. (14) can be decomposed on the first-order operators $\hat{A}$ and $\hat{A}^\dagger$ appearing in Eqs. (8) and (9), whose ground state solutions describes Gompertzian growth and regression

$$\hat{A}\psi(q)=\frac{1}{\sqrt{2}}\left[\frac{d}{dt}+W(t)\right]\psi(q)=0,$$
$$\hat{A}^\dagger\psi(q)_G^\dagger=\frac{1}{\sqrt{2}}\left[-\frac{d}{dt}+W(t)\right]\psi(q)_G^\dagger=0, \tag{15}$$

here $W=-x(q)=-\exp(-q)$. In the same manner, one may construct the second and first order equations for the growth and regression of WBE-type systems by taking advantage the superpotential W calculated from Eq. (6) and explicit forms of the WBE growth and regression functions (3) and (7).

3.3. Quantum oscillatory phenomena

The quantity $x(q)$ enables to associate via Eq. (8) a potential energy $V(q)$ and eigenvalue ϵ with all types of functions $\psi(q)$ derived in CDG scheme. On the other hand, $\psi(q)$ is interpreted as the solution of the differential equation (8), whose form resembles the quantal Schrödinger formula. To prove that the CDG approach produces not only classical (macroscopic) growth functions but also quantal (microscopic) once, one may apply a linear expansion of the generating function $\Phi(x)=c_0+c_1x$, which includes a constant term $c_0\neq 0$ omitted in the CDG scheme and c_1 coefficient, which in the original CDG approach was constrained to 1 [1]. This CDG generalization was proposed first by Molski [19] and then employed in investigations of the classical oscillations and growth-regression states in [17,18]. To generate the quantal solutions in the CDG scheme we assume that $x(0)=(1-c_0)/c_1$, which for $c_0=0$, $c_1=1$ gives the CDG initial condition $x(0)=1$. Employing Eqs. (1) and (3) by integration one gets $(c_0, c_1>0)$

$$x(q)=\frac{1}{c_1}\left[\exp(-c_1q)-c_0\right],$$
$$\psi(q)=\exp\left\{\frac{1}{c_1^2}\left[1-\exp(-c_1q)\right]\right\}\exp\left(-\frac{c_0}{c_1}q\right), \tag{16}$$

which by making use of correspondences $c_1^2=2x_e$, $c_0=1-x_e$ can be converted to alternative forms

$$x(q)=\left[\frac{\exp(-\sqrt{2x_e}q)-1+x_e}{\sqrt{2x_e}}\right],$$
$$\psi(q)_0=\exp\left[\frac{1-\exp\left(-\sqrt{2x_e}q\right)}{2x_e}\right]\exp\left[\frac{(x_e-1)q}{\sqrt{2x_e}}\right]. \tag{17}$$

Taking advantage of the Riccati equation (8) one may derive the second order differential equation whose solution is function (17)

$$\left\{-\frac{1}{2}\frac{d^2}{dq^2}+\frac{1}{4x_e}\left[1-\exp(-\sqrt{2x_e}q)\right]^2-\epsilon_0\right\}\psi(q)_0=0, \tag{18}$$

which includes eigenvalue $\epsilon_0=1/2-x_e/4$ being a ground state $(v=0)$ version of a general formulae (in dimensionless unit) $\epsilon_v=v+1/2-x_e(v+1/2)^2$, $v=0,1,2\dots$ It is interesting to note that Eq. (18) under substitutions $\epsilon_0=E_0$, $x_e=\hbar\omega/4D_e$, $\omega=a\sqrt{2D_e/m}$ and $q=a(r-r_0)/\sqrt{2x_e}$ converts to the Schrödinger equation for the ground state of Morse oscillator [29]

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+D_e\left[1-\exp\left[-a(r-r_0)\right]\right]^2-E_0\right\}\psi(r)_0=0, \tag{19}$$

whereas for $\epsilon_0=P_0c$, $x_e=\hbar\omega/4D_e$, $\omega=a\sqrt{2D_e/mc^2}$ and $q=a(t-t_0)/\sqrt{2x_e}$ Eq. (18) yields the quantal Horodecki equation [31] for the ground state of time-dependent Morse oscillator [28]

$$\left\{-\frac{\hbar^2}{2mc^2}\frac{d^2}{dt^2}+D_e\left[1-\exp\left[-a(t-t_0)\right]\right]^2-P_0c\right\}\psi(t)_0=0. \tag{20}$$

Eq. (20) represents the non-relativistic version of the relativistic Feinberg equation [30], first time derived by Horodecki [31]; it is a space-like (nonlocal) counterpart of the time-like (local) Schrödinger formula (19). In Eqs. (19) and (20) x_e is anharmonicity constant, ω – frequency, D_e – dissociation constant, m – mass, c – light velocity. The eigenvalues $E_0=(\hbar\omega)(1/2-x_e/4)$ and $P_0=(\hbar\omega/c)(1/2-x_e/4)$ for $x_e=0$ reduces to $E_0=\hbar\omega/2$ and $P_0=\hbar\omega/2c$ valid for the ground state of space- and time-dependent harmonic oscillators. In this case, which corresponds to $c_1=0$ and $\Phi(x)=c_0=1$, Eqs. (16) convert to $x(q)=-q$ and $\psi(q)_0=\exp\left(-q^2/2\right)$ representing a ground state of the quantal harmonic oscillator. Hence, E_0 and P_0 can be interpreted as the zero-point energy and momentum of vacuum [32,33], which are consequences of the random spatial (temporal) fluctuations of vacuum and position-momentum (time-energy) uncertainty principle.

It is worth emphasizing that the time-depended version of $\psi(q)$ specified by Eq. (16) for $c_0, c_1<0$ reproduces Makeham function [34,35], which has been used in actuarial science for specifying a simplified mortality law. In this connection $\psi(q)$ represents the probability that a newborn will achieve age q.

Proceeding in the same manner as for the first term of Eq. (2) one can derive $x(q)$ and $\psi(q)$ for the second order expansion of $\Phi(x)=c_1(x+c_0/c_1)+c_2(x+c_0/c_1)^2$ and identical as before initial condition $x(0)=(1-c_0)/c_1$

$$\begin{aligned}
x(q)&=\frac{c_1\exp(-c_1q)}{c_1^2+c_2-c_2\exp(-c_1q)}-\frac{c_0}{c_1}\\
&=\frac{(sc_1/c_2)\exp[-c_1(q-q_0)]}{1-s\exp[-c_1(q-q_0)]}-\frac{c_0}{c_1},\\
\psi(q)&=\{1-s\exp[-c_1(q-q_0)]\}^{1/c_2}\{s\exp[-c_1(q-q_0)]\}^{c_0/c_1^2},\\
q_0&=\frac{1}{c_1}\ln\left[\frac{2c_1^2-c_1^2c_2+2c_0c_2}{(2c_0+c_1^2)(c_1^2+c_2)}\right],\\
s&=\frac{c_2(2c_0+c_1^2)}{2c_1^2-c_1^2c_2+2c_0c_2},
\end{aligned} \tag{21}$$

and then one may construct the quantal Schrödinger and Feinberg–Horodecki equations

$$\left\{-\frac{1}{2}\frac{d^2}{dq^2}+D\left[\frac{1-\exp[-c_1(q-q_0)]}{1-s\exp[-c_1(q-q_0)]}\right]^2-\epsilon\right\}\psi(q)=0, \tag{22}$$

for the ground state of the time-dependent Wei oscillator [21] in which $D=(2c_0+c_1^2)^2/8c_1^2(1-c_2)$ and $\epsilon=D-c_0^2/2c_1^2$. The solution (21) and parameters appearing in (22) can be specified in the form applied by Wei [21] employing replacements $c_1=b$, $s=c$, $1/c_2=\rho+1/2$, $c_0/c_1^2=\rho_0$.

It should be pointed out here that only modified form of the generated function (2) produces quantal solution (21) for the Wei oscillator, whereas the conventional form $\Phi(x)=c_0+c_1x+c_2x^2$ applied by CDG leads to the classical solutions for the dumped oscillations first time derived in [17].

The results obtained in this section clearly indicate that the class of PU can be extended to include quantum oscillatory phenomena for harmonic as well as anharmonic Morse and Wei oscillators generating subclasses $Q0$, $Q1$ and $Q2$ of PU for different powers $n=0,1,2$ of the truncated series (2) for $c_0\neq 0$. In the next section it will be demonstrated that the quantal subclasses Qn of PU are generalization of the classical Un due to the transformation $Qn\xrightarrow{c_0=0}Un$ taking place for $n=1,2$.

3.4. Coherence

Continuing the search for further analogies between the CDG and quantum formalisms, we find that equations of growth (1) and regression (5) can be specified for $\alpha=\alpha^*=0$, $|0\rangle=\psi(q)$ and $\langle 0|=\psi(q)^\dagger$ in the forms

$$\begin{aligned}
&\hat{A}|\alpha\rangle=\alpha\psi(q)\exp[\sqrt{2}\alpha q],\qquad \langle\alpha|\hat{A}^\dagger=\alpha^*\psi(q)^\dagger\exp(\sqrt{2}\alpha^*q),\\
&\hat{A}=\frac{1}{\sqrt{2}}\left[\frac{d}{dq}-x(q)\right],\quad \hat{A}^\dagger=\frac{1}{\sqrt{2}}\left[-\frac{d}{dq}-x(q)\right],\\
&[\hat{A},\hat{A}^\dagger]=\Phi(x)
\end{aligned} \tag{23}$$

familiar in the quantum theory of minimum uncertainty coherent states of harmonic and anharmonic oscillators [27]. The coherent states, which minimize the generalized position-momentum [29] (local states) or time-energy [28] (nonlocal states) uncertainty relations are eigenstates of the annihilation operator $\hat{A}$. They not only minimize the Heisenberg relations, but also maintain those relations in time (space) due to their temporal (spatial) stability, hence they are called *intelligent* coherent states [37]. One may prove that coherent states (23) minimize the generalized uncertainty relation ($\hbar=1$) [29,28]

$$[\Delta x(q)]^2(\Delta\epsilon)^2\geq\frac{1}{4}\langle\alpha|\Phi(x)|\alpha\rangle^2,\ \Phi(x)=\mp i\left[x(q),\hat{\epsilon}\right], \tag{24}$$

in which $\hat{\epsilon}=\pm id/dq$ represents energy (+) or momentum (−) operator whereas $x(q)$ plays the role of a temporal (spatial) anharmonic variable associated with a given potential. To carry out the proof, the following relationships should be derived for normalized states $\langle\alpha||\alpha\rangle=1$

$$\begin{aligned}
\langle\alpha|x(q)|\alpha\rangle&=-\frac{1}{\sqrt{2}}\langle\alpha|\hat{A}+\hat{A}^\dagger|\alpha\rangle=-\frac{1}{\sqrt{2}}\left(\alpha+\alpha^*\right),\\
\langle\alpha|\hat{\epsilon}|\alpha\rangle&=i\frac{1}{\sqrt{2}}\langle\alpha|\hat{A}-\hat{A}^\dagger|\alpha\rangle=i\frac{1}{\sqrt{2}}\left(\alpha-\alpha^*\right),\\
2\langle\alpha|x(q)^2|\alpha\rangle&=\left(\alpha+\alpha^*\right)^2+\langle\alpha|\Phi(x)|\alpha\rangle,\\
-2\langle\alpha|\hat{\epsilon}^2|\alpha\rangle&=\left(\alpha-\alpha^*\right)^2-\langle\alpha|\Phi(x)|\alpha\rangle.
\end{aligned} \tag{25}$$

Having derived Eqs. (25) we can calculate the squared standard deviations

$$\begin{aligned}
\Delta x(q)^2&=\langle\alpha|x(q)^2|\alpha\rangle-\langle\alpha|x(q)|\alpha\rangle^2=\frac{1}{2}\langle\alpha|\Phi(x)|\alpha\rangle,\\
\Delta\epsilon^2&=\langle\alpha|\hat{\epsilon}^2|\alpha\rangle-\langle\alpha|\hat{\epsilon}|\alpha\rangle^2=\frac{1}{2}\langle\alpha|\Phi(x)|\alpha\rangle,
\end{aligned} \tag{26}$$

which prove that

$$[\Delta x(q)]^2(\Delta\epsilon)^2=\frac{1}{4}\langle\alpha|\Phi(x)|\alpha\rangle^2. \tag{27}$$

Eq. (27) is satisfied both for $\alpha\neq 0$ as well as $\alpha=0$ and an arbitrary form of generating function $\Phi(x)$. Those facts indicate that $\psi(q)$ in CDG approach can be interpreted as a minimum uncertainty coherent state of oscillator characterized by anharmonic variable $x(q)$. It is noteworthy that this interpretation remains independent of the type of generating function $\Phi(x)$, hence it can be applied both to micro- and macroscopic systems, characterized by $c_0\neq 0$ and $c_0=0$, respectively. In particular, using the CDG approach one may construct the coherent states of the time- and space-dependent Morse oscillator, which for $c_0=0$ and $c_1=1$ convert to the Gompertzian coherent states of growth (regression) first time derived by Molski and Konarski [38]

$$\frac{1}{\sqrt{2}}\left\{\frac{d}{dq}-\frac{1}{c_1}[\exp(-c_1q)-c_0]\right\}|\alpha\rangle =$$
$$\alpha\exp\left\{\frac{1}{c_1^2}[1-\exp(-c_1q)]\right\}\exp\left[-\frac{c_0q}{c_1}\right]\exp(\sqrt{2}\alpha q)\overset{c_0=0,c_1=1}{\longrightarrow}$$
$$\frac{1}{\sqrt{2}}\left\{\frac{d}{dq}-[\exp(-q)]\right\}|\alpha\rangle =$$
$$\alpha\exp[1-\exp(-q)]\exp(\sqrt{2}\alpha q)\overset{\alpha=0}{\longrightarrow}$$
$$\left[\frac{d}{dq}-\exp(-q)\right]\exp[1-\exp(-q)]=0, \tag{28}$$

$$\langle\alpha|\frac{1}{\sqrt{2}}\left\{-\frac{d}{dq}-\frac{1}{c_1}[\exp(-c_1q)-c_0]\right\} =$$
$$\alpha^*\exp\left\{-\frac{1}{c_1^2}[1-\exp(-c_1q)]\right\}$$
$$\times\exp\left[\frac{c_0q}{c_1}\right]\exp(\sqrt{2}\alpha^* q)\overset{c_0=0,c_1=1}{\longrightarrow}$$
$$\langle\alpha|\frac{1}{\sqrt{2}}\left\{-\frac{d}{dq}-[\exp(-q)]\right\} =$$
$$\alpha^*\exp\{-[1-\exp(-q)]\}\exp(\sqrt{2}\alpha^* q)\overset{\alpha^*=0}{\longrightarrow}$$
$$\left[-\frac{d}{dq}-\exp(-q)\right]\exp\{-[1-\exp(-q)]\}=0. \tag{29}$$

In a similar manner, one may construct the coherent states of time- and space-dependent Wei oscillator, which in the dissociation (classical) limit convert to the coherent WBE-type function of growth

$$\frac{1}{\sqrt{2}}\left\{\frac{d}{dq}-\frac{(sc_1/c_2)\exp[-c_1(q-q_0)]}{1-s\exp[-c_1(q-q_0)]}+\frac{c_0}{c_1}\right\}|\alpha\rangle =$$
$$\alpha\{1-s\exp[-c_1(q-q_0)]\}^{1/c_2}\{s\exp[-c_1(q-q_0)]\}^{c_0/c_1^2}$$
$$\times\exp(\sqrt{2}\alpha q)\overset{c_0=0,c_1=1}{\longrightarrow}$$
$$\frac{1}{\sqrt{2}}\left\{\frac{d}{dq}-\frac{(s'/c_2)\exp\left[-(q-q_0')\right]}{1-s'\exp\left[-(q-q_0')\right]}\right\}|\alpha\rangle =$$
$$\alpha\left\{1-s'\exp\left[-(q-q_0')\right]\right\}^{1/c_2}\exp(\sqrt{2}\alpha q)\overset{\alpha=0}{\longrightarrow}$$
$$\left\{\frac{d}{dq}-\frac{(s'/c_2)\exp\left[-(q-q_0')\right]}{1-s'\exp\left[-(q-q_0')\right]}\right\}\left\{1-s'\exp\left[-(q-q_0')\right]\right\}^{1/c_2} =$$
$$\left[\frac{d}{dq}-\frac{\exp(-q)}{1+c_2-c_2\exp(-q)}\right][1+c_2-c_2\exp(-q)]^{1/c_2}=0. \tag{30}$$

Here $q_0'=\ln[(2-c_2)/(1+c_2)]$ and $s'=c_2/(2-c_2)$. Analogically the WBE states of regression can be derived from quantal solutions of the creation equation (23).

3.5. The Fokker–Planck equation

The combination of the CDG approach with SUSYQM formalism seems to be a convenient tool for generating analytical solutions of the Fokker–Planck (FP) equation [39]

$$\frac{\partial}{\partial\tau}P(q,\tau)=-\frac{\partial}{\partial q}[f(q)P(q,\tau)]+\frac{\Gamma}{2}\frac{\partial^2}{\partial q^2}P(q,\tau) \tag{31}$$

specified in dimensionless spatial q and temporal τ coordinates. In Eq. (31) $\Gamma/2$ and $f(q)$ represent dimensionless diffusion constant and the drift term related to the potential energy of the system, whereas $P(q,\tau)$ describes the probability distribution function. The FP equation is used to study stochastic phenomena e.g. Brownian motions and other diffusion governed processes. Their nature requires application of probabilistic techniques to describe unpredictable paths emerging in the complex system under influence of thermal fluctuations, mechanical collisions as well as physical and chemical interactions between its micro-components. There are numerous practical applications of the FP equation in domain of physics, chemistry, biology, astronomy, economy and other sciences [39], which are based on solutions of Eq. (31) generated by numerical integration, simulation methods or analytical procedures. In the latter case the probability function $P(q,\tau)$ can be specified in the form [40,41]

$$P(q,\tau)=\sum_{\nu=0}^{\infty}a_\nu\psi(q)_0\psi(q)_\nu\exp(-\tau|\lambda_\nu|),\quad a_\nu=\frac{\psi(0)_0}{\psi(0)_\nu} \tag{32}$$

which permits decomposition of (31) onto the set of Schrödinger-like equations

$$\left\{-\frac{\Gamma}{2}\frac{\partial^2}{\partial q^2}+\frac{1}{2}\left[\frac{f(q)^2}{\Gamma}+\frac{df(q)}{dq}\right]-\lambda_\nu\right\}\psi(q)_\nu=0,$$
$$\nu=0,1,2\ldots \tag{33}$$

which can be written in the factorized form $(\hat{F}^\dagger\hat{F}-\lambda_\nu)\psi(q)_\nu=0$

$$\hat{F}^\dagger=\frac{1}{\sqrt{2}}\left[-\sqrt{\Gamma}\frac{d}{dq}-f(q)/\sqrt{\Gamma}\right]$$
$$\hat{F}=\frac{1}{\sqrt{2}}\left[\sqrt{\Gamma}\frac{d}{dq}-f(q)/\sqrt{\Gamma}\right] \tag{34}$$

amenable to straightforward analytical treatment in the CDG scheme. To this aim, Eqs. (1), (2) and (3) are converted to the forms adequate to the FP formalism

$$\hat{F}\psi(q)_0=0,\qquad -\frac{df(q)}{dq}=\Phi(f)=c_0+c_1f(r)+\ldots,$$
$$\psi(q)_0=\exp\left[\Gamma^{-1}\int\limits_q f(q)dq+C\right]. \tag{35}$$

In the simplest case of $Q0$ subclass of PU characterized by $\Phi(f)=c_0\neq 0$ from (35) and condition $\psi(0)=1$ one gets $f(q)=-c_0q$ and the ground state solution $\psi(q)_0=\exp([-c_0q^2/(2\Gamma)]$ satisfying Eq. (33) for $\lambda_0=0$. Eq. (33) can be specified in an alternative form

$$\left(-\frac{\Gamma}{2}\frac{\partial^2}{\partial q^2}+\frac{c_0^2}{2\Gamma}q^2-\frac{c_0}{2}-\lambda_\nu\right)\psi(q)_\nu=0, \tag{36}$$

by calculating the Riccati's term appearing in (33) and defined by (8). The complete set of analytical solutions of Eq. (36) takes the well-known form

$$\Psi(q)_\nu=\frac{1}{\sqrt{2^\nu\nu!}}\left(\frac{c_0}{\Gamma\pi}\right)^{1/4}\exp\left[-c_0q^2/(2\Gamma)\right]H_\nu(\sqrt{c_0/\Gamma}q) \tag{37}$$

being a classical counterpart of the quantal harmonic oscillator solutions associated with eigenvalues $\lambda_\nu=c_0\nu$ and physicists' Hermite polynomials $H_\nu(q)$.

Having determined the eigenfunctions (37) and eigenvalues λ_ν, we can calculate coefficients a_ν appearing in (32) and construct the final form of the probability function for $Q0$ class of solutions of the FK equation

$$P(q,\tau)=\sum_{\nu=0}^{\infty}\frac{1}{H_\nu(0)}\left(\frac{c_0}{\Gamma\pi}\right)^{1/2}\exp\left(-c_0q^2/\Gamma\right)$$
$$\times H_\nu\left(\sqrt{c_0/\Gamma}q\right)\exp(-\tau c_0\nu). \tag{38}$$

For $\tau \to \infty$ Eq. (38) reduces to the well-known classical formula

$$P(q) = [c_0/(\Gamma\pi)]^{1/2} \exp(-c_0 q^2/\Gamma)$$
$$= P(r) = [c_0/(\Gamma\pi)]^{1/2} \exp[-c_0 u_r^2 (r - r_0)^2/\Gamma] \quad (39)$$

representing stationary state of the Ornstein–Uhlenbeck process identified with the Gaussian (normal) distribution [39]. Here u_r is a scaling factor in units $[m^{-1}]$, r_0 is a mean or expectation of the distribution, $\Gamma/(c_0 u_r^2) = \sigma^2$ stands for variance and σ is standard deviation of the mean.

Proceeding in the same manner as for $Q0$ class of PU and using constraints $\Psi(0) = 1$ and $f(0) = (1 - c_0)/c_1$ we can solve Eqs. (35) for $Q1$ class represented by $\Phi(f) = c_0 + c_1 f(q)$ providing $f(q) = (1/c_1)[\exp(-c_1 q) - c_0]$ and the FP equation including the generalized ($k \neq 1$) [36] or standard ($k = 1$) [20] Morse function, amenable to direct analytical treatment

$$\left\{ -\frac{\Gamma}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2c_1^2\Gamma}[k - \exp(-c_1 q)]^2 - \frac{k}{2} + \left(\frac{c_1^2\Gamma}{2}\right)\frac{1}{4} - \lambda_\nu \right\}$$
$$\times \psi(q)_\nu = 0. \quad (40)$$

The solutions and associated eigenvalues take the form

$$\psi(q)_\nu = N_\nu \exp\left\{ \frac{1}{\Gamma c_1^2}[1 - \exp(-c_1 q)] \right\}$$
$$\times \exp\left[-c_1 q \left(\frac{k}{c_1^2\Gamma} - \nu - \frac{1}{2} \right) \right]$$
$$L_\nu^{[2/(c_1^2\Gamma) - 2\nu - 1]}\left[2/(c_1^2\Gamma)\exp(-c_1 q) \right],$$
$$N_\nu = \sqrt{\frac{c_1[2/(c_1^2\Gamma) - 2\nu - 1]\nu!}{\Gamma[2/(c_1^2\Gamma) - \nu]}},$$
$$\lambda_\nu = k\nu - \frac{c_1^2\Gamma}{2}(\nu^2 + \nu) = c_0\nu - \frac{c_1^2\Gamma}{2}\nu^2, \quad \nu = 0, 1, 2 \ldots$$
$$k = c_0 + \frac{c_1^2\Gamma}{2}. \quad (41)$$

Here, $L_\nu^\alpha(q)$ and $\Gamma(\beta)$ represent generalized Laguerre polynomials and gamma function, respectively and Eq. (40) can be interpreted as a classical counterpart of the wave equation for the generalized Morse potential $V(q) = D_e[k - \exp(-c_1 q)]^2$.

Introducing $\psi(q)_\nu$, $\psi(0)_\nu$, N_ν, and $\lambda_\nu = k\nu - c_1^2\Gamma/2(\nu^2 + \nu)$ into Eq. (32) one gets the probability distribution function $P(q, \tau)$, which for $\tau \to \infty$ reduces to the stationary formula

$$P(q) = N_0^2 \exp\left\{ \frac{2}{\Gamma c_1^2}[1 - \exp(-c_1 q)] \right\}$$
$$\times \exp\left[-2c_1 q \left(\frac{k}{c_1^2\Gamma} - \frac{1}{2} \right) \right] = P(r) =$$
$$N_0^2 \exp\left\{ \frac{2}{\Gamma c_1^2}\{1 - \exp[-c_1 u_r (r - r_0)]\} \right\}$$
$$\times \exp\left[-2c_1 u_r (r - r_0) \left(\frac{k}{c_1^2\Gamma} - \frac{1}{2} \right) \right] \quad (42)$$

representing asymmetric, right-skewed distribution function (see Fig. 1). Its form differs from the normal distribution (39) endowed with the null skewness. However, in reality, some data points may not be perfectly symmetric, so the skewness of the distribution functions correctly reproducing the data sets is an important property worth considering for the sake of practical applications. It should be pointed out also that skewed function (42) is a generalization of the sigmoidal Gompertz function (4) to be derived from (42) by substitutions $c_1^2\Gamma = 2$ and $u_r = c_1 = k = 1$. In the same manner as for $Q0$ and $Q1$ classes one may calculate $P(q, \tau)$ and $P(q)$ for $Q2$ class of PU including processes governed by the drift term $f(q)$ associated with the anharmonic Wei potential [21]. The results obtained in this section can be applied in modeling of biological growth and mechanisms involving cells differentiation with possible applications in determining optimal strategy for the medical treatment [41] (and references cited therein).

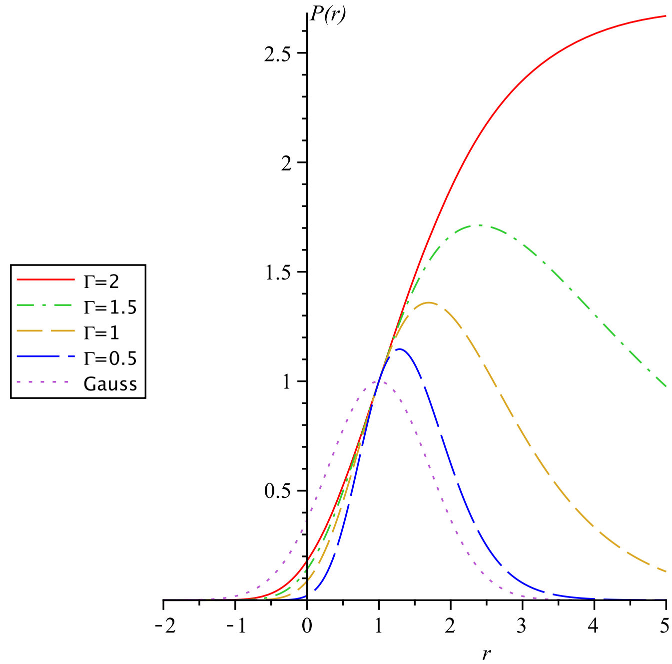

Fig. 1. The plots of the non-normalized distribution function $P(r)$ (42) for $u_r = c_1 = r_0 = k = 1$ and different values of the diffusion constant: $\Gamma = 2$, $\Gamma = 1.5$, $\Gamma = 1$, $\Gamma = 0.5$. The solid and dot plots present Gompertz ($\Gamma = 2$) $P(r) = \exp[1 - \exp(-r + 1)]$ and Gauss $P(r) = \exp[-(r - 1)^2]$ distributions, respectively.

4. Conclusions

The results obtained indicate that the concept of PU originally applied only to macroscopic complex systems can be extended to include quantum oscillatory phenomena, coherence and supersymmetry playing a vital role on the microscopic level. In connection presented a micro-macro conversion is accomplished by $c_0 \to 0$, which transforms quantum equations into classical ones. Only one exception is quantal uncertainty relation (24), which is satisfied both for micro- and macroscopic functions $\psi(q)$ generated for an arbitrary form of $\Phi(x)$. This fact has very important interpretative implications. The ordinary time-like coherent states, which minimize the position-momentum uncertainty relation evolve coherently in time being localized on the classical space-trajectory [42], on the contrary, the space-like coherent states which minimize the time-energy uncertainty relation evolve along localized (classical) time-trajectory being coherent in all points of space [28,38]. Such states assumed to be coherent at an arbitrary point of space remain coherent in all points of space. We conclude that the spatial coherence is an immanent feature of all systems whose growth (decay) is described by time-dependent functions derived in the CDG scheme independently of their quantal or classical nature. Although the notions of coherence and supersymmetry are usually attributed to microscopic systems, the correspondence principle introduced by Niels Bohr [43] allows for the physical characteristics of quantum systems to be maintained also in classical regime.

According to this concept, the quantum theory of micro-objects passes asymptotically into the classical one when the quantum numbers characterizing the micro-system attain extremely high values or we can neglect the Planck's constant. In this way one may derive e.g. from quantal Planck's black-body radiation formula the classical Rayleigh–Jeans law describing the spectral radiance of electromagnetic waves. Both models describe the same phenomenon but employ diverse (quantum vs classical) formalisms and are valid for different wavelength ranges of emitted radiation. Identical situation appears in the case of quantal oscillatory phenomena which in the classical limit possess the same characteristics as their quantum counterparts. The first- and second-order growth equations obtained in this way do not contain mass nor Planck's constant [28,38], therefore according to the correspondence principle, they represent classical equations of coherent growth (regression). It is straightforward to demonstrate that for $c_0 = 0$ quantal Eqs. (19), (20), (28), (29) and (30) convert to their classical counterparts characterized by the dissociation condition $\epsilon = D$. We conclude that the macroscopic Gompertz and WBE-type functions have identical forms as microscopic ground state solutions of the quantal equation for the time-depended Morse and Wei oscillators in the dissociation state. Consequently, the quantal subclasses Qn of PU are generalization of the classical Un introduced by CDG [1] due to the relationship $Qn \xrightarrow{c_0=0} Un$ taking place for $n = 1, 2$. It should be pointed out that in the dissociation limit (or $c_0 = 0$) the direction of temporal growth (regression) is consistent with the arrow of time, hence it is not of the oscillatory type as predicted for hypothetical bound states of time-dependent oscillators. The extension of the PU strategy presented in this work permits including in the CDG classification scheme not only quantum oscillatory phenomena belonging to $Q0$, $Q1$ and $Q2$ subclasses of PU, but also the coherence and supersymmetry proving that they persist both in micro- as well as macro domains. Hence, the results obtained reveal existence of a new class (according to the Leggett classification [44]) of macroscopic quantum (or quasi-quantum) phenomena, which may play a vital role in coherent (local and nonlocal) formation of the specific growth patterns in complex systems. The extended CDG formalism including the space-dependent phenomena can be applied to generate the coherent states of the space-dependent Morse and Wei oscillators, which minimize the position-momentum uncertainty relation [36] and in dissociation limit $c_0 \to 0$ or, equivalently $E \to D$, reduce to the space-dependent sigmoidal Gompertz and WBE-like functions widely applied in a range of fields including e.g. probability theory and statistics where they are used to describe cumulative distribution of entities characterized by different spatial sizes [24]. A combination of the CDG and SUSYQM theories provides a convenient tool for generating analytical solutions of the FP equation with the drift term $f(q)$ related to the pseudopotential $-W(q)$ to be determined for harmonic as well as anharmonic Morse and Wei potentials. The exact solutions of the FP equation obtained in this way can be used in determining the probability distribution function $P(q, \tau)$ widely used in many areas of sciences [39] including modeling of the processes in biological systems and mechanisms involving cells proliferation and differentiation with possible application in evaluating optimal strategies for the medical treatment [45,46].

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